A Counterexample to the Hertz–Jabbour–Cerf Concavity Assumption

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Abstract. Hertz, Jabbour, and Cerf proposed and numerically tested the concavity of a covariance-corrected entropic uncertainty functional. We show that this functional is not even midpoint concave on the set of one-mode quantum states. The two endpoint states are centered Gaussian states; their equal mixture is non-Gaussian. The proof is analytic and uses only the one-mode covariance physicality condition and the elementary entropy-of-mixtures inequality. This invalidates the full concavity assumption used in their conditional argument, but does not address the truth of the motivating entropy-power uncertainty inequality itself.

1. The concavity assumption

Work in units in which $\hbar = 1$, so that $[X, P] = i$, and let all logarithms be natural. Put $R = (X, P)^{\mathsf{T}}$, $d_j = \mathrm{Tr}(\rho R_j)$, and use the symmetrized covariance convention

$$(\gamma_\rho)_{jk} := \frac{1}{2}\,\mathrm{Tr}\big[\rho\,\{R_j - d_j, R_k - d_k\}\big]\,.$$

For a one-mode state ρ in the natural domain of F, write $h_\rho(X)$ and $h_\rho(P)$ for the differential entropies of its position and momentum marginals. Thus

$$\gamma_\rho = \begin{pmatrix} \sigma_X^2(\rho) & \sigma_{XP}(\rho) \\ \sigma_{XP}(\rho) & \sigma_P^2(\rho) \end{pmatrix}$$

is its covariance matrix. Hertz, Jabbour, and Cerf [1] define

$$F(\rho) := h_\rho(X) + h_\rho(P) - \frac{1}{2}\log\left(\frac{\sigma_X^2(\rho)\,\sigma_P^2(\rho)}{\det\gamma_\rho}\right). \tag{1}$$

They assume, and numerically test, that for states ρ_1, ρ_2 in this domain and $\lambda \in [0, 1]$,

$$F(\lambda\rho_1 + (1-\lambda)\rho_2) \geq \lambda F(\rho_1) + (1-\lambda)F(\rho_2). \tag{2}$$

The original paper proves only a special case in which the two states have the same covariance matrix [1, Appendix A.3]. The general statement is retained as Assumption (ii) in the later review by Hertz and Cerf [2, Sec. 3.4].

For a centered Gaussian state ρ_γ with covariance γ, the two marginals are Gaussian. Hence

$$F(\rho_\gamma) = \log(2\pi e) + \frac{1}{2}\log\det\gamma. \tag{3}$$

2. An exact Gaussian counterexample

Main result. There exist two physical centered one-mode Gaussian states ρ_0 and ρ_M such that

$$F\left(\frac{\rho_0 + \rho_M}{2}\right) < \frac{F(\rho_0) + F(\rho_M)}{2}.$$

Consequently, F is not concave.

Theorem 1. *For every $M > 4097/4$, consider centered Gaussian states with covariance matrices*

$$\gamma_0 = \frac{1}{2}I_2, \qquad \gamma_M = \begin{pmatrix} M & M - \frac{1}{2} \\ M - \frac{1}{2} & M \end{pmatrix}. \tag{4}$$

Both matrices are physical, and the corresponding states violate (2) at $\lambda = \frac{1}{2}$. In particular, $M = 4096$ gives an explicit example with integer parameter M.

Proof. Writing $\Omega = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the one-mode physicality condition is

$$\gamma + \frac{i}{2}\Omega \succeq 0 \quad \Longleftrightarrow \quad \gamma \succ 0 \ \text{ and } \ \det\gamma \geq \frac{1}{4}$$

under the convention above, after rescaling the quadrature convention used in [3]. Here

$$\det\gamma_0 = \frac{1}{4}, \qquad \det\gamma_M = M - \frac{1}{4},$$

while the eigenvalues of γ_M are $\frac{1}{2}$ and $2M - \frac{1}{2}$. Thus both matrices in (4) define valid Gaussian states. Equation (3) gives

$$F(\rho_0) = \log(\pi e), \tag{5}$$

$$F(\rho_M) = \log(2\pi e) + \frac{1}{2}\log\left(M - \frac{1}{4}\right). \tag{6}$$

Set $\bar\rho := \frac{1}{2}(\rho_0 + \rho_M)$. Because both states are centered, its covariance matrix is

$$\bar\gamma = \frac{\gamma_0 + \gamma_M}{2} = \begin{pmatrix} a & c \\ c & a \end{pmatrix}, \qquad a = \frac{M + \frac{1}{2}}{2}, \quad c = \frac{M - \frac{1}{2}}{2}. \tag{7}$$

In particular,

$$\det\bar\gamma = (a-c)(a+c) = \frac{M}{2}. \tag{8}$$

Each position or momentum marginal of $\bar\rho$ has density

$$q = \frac{1}{2}\phi_{1/2} + \frac{1}{2}\phi_M, \qquad \phi_v(t) := \frac{1}{\sqrt{2\pi v}}\exp\left(-\frac{t^2}{2v}\right), \tag{9}$$

Substituting (7)–(9) into (1) yields

$$F(\bar\rho) = 2h(q) - \log a + \frac{1}{2}\log\frac{M}{2}. \tag{10}$$

Let $Z \sim \text{Bernoulli}(\frac{1}{2})$ select the component in (9), and let the classical random variable $Y \mid Z$ have the selected Gaussian law. The identity $h(Y) = h(Y \mid Z) + I(Y;Z)$, together with $I(Y;Z) \leq H(Z) = \log 2$, gives

$$h(q) \leq \log 2 + \frac{1}{2}h(\phi_{1/2}) + \frac{1}{2}h(\phi_M). \tag{11}$$

Since $h(\phi_v) = \frac{1}{2}\log(2\pi e v)$, equations (10) and (11) imply the explicit analytic bound

$$F(\bar\rho) \leq \log(\pi e) + \log\left(\frac{16M}{2M+1}\right). \tag{12}$$

On the other hand, (5)–(6) give

$$\frac{F(\rho_0) + F(\rho_M)}{2} = \log(\pi e) + \frac{1}{2}\log 2 + \frac{1}{4}\log\left(M - \frac{1}{4}\right). \tag{13}$$

The upper bound in (12) is strictly smaller than (13) whenever

$$\left(\frac{16M}{2M+1}\right)^4 < 4M - 1. \tag{14}$$

For every $M > 4097/4$, the fraction $16M/(2M+1)$ is smaller than 8, and

$$\left(\frac{16M}{2M+1}\right)^4 < 8^4 = 4096 < 4M - 1.$$

Thus (14) holds, proving the strict reverse of midpoint concavity. □

3. Scope, priority, and AI assistance

The conditional argument in [1, 2] uses both a pure-state minimization claim and concavity of F. The present construction invalidates only the full concavity premise as written. Since ρ_M is mixed, it does not rule out a weaker Jensen property restricted to pure-state ensembles. It neither disproves the proposed entropy-power uncertainty inequality nor resolves the separate Wigner-entropy conjecture; the equal-covariance special case also remains intact.

Priority caveat. As of 1 August 2026, citation-chain and exact-formula searches found no prior indexed proof, counterexample, or matching construction. This is not proof of priority: unindexed manuscripts, correspondence, and historical social-media posts may have been missed. We make no claim of being first pending author contact and independent expert review.

Provenance and verification. This working note records a counterexample identified and initially drafted with substantial assistance from OpenAI Codex in ChatGPT (GPT-5.6 Sol), including literature searching and algebraic checking. These checks were AI-assisted and do not constitute independent expert verification; correctness and priority remain subject to review.

References

[1] A. Hertz, M. G. Jabbour, and N. J. Cerf, "Entropy-power uncertainty relations: towards a tight inequality for all Gaussian pure states," *Journal of Physics A: Mathematical and Theoretical* **50** (2017), 385301. doi:10.1088/1751-8121/aa852f; arXiv:1702.07286.

[2] A. Hertz and N. J. Cerf, "Continuous-variable entropic uncertainty relations," *Journal of Physics A: Mathematical and Theoretical* **52** (2019), 173001. doi:10.1088/1751-8121/ab03f3; arXiv:1809.01052.

[3] C. Weedbrook, S. Pirandola, R. García-Patrón, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, "Gaussian quantum information," *Reviews of Modern Physics* **84** (2012), 621–669. doi:10.1103/RevModPhys.84.621; arXiv:1110.3234.